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A semigroup proof of the bounded degree case of S.B. Rao's Conjecture on degree sequences and a bipartite analogue

Christian Joseph Altomare

The Ohio State University, 231, West 18th Avenue, Columbus, OH, United States

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ABSTRACT

S.B. Rao conjectured in 1980 that graphic degree sequences are well quasi ordered by a relation $\preccurlyeq$ defined in terms of the induced subgraph relation (Rao, 1981 [7]). In 2008, M. Chudnovsky and P. Seymour proved this longstanding conjecture by giving structure theorems for graphic degree sequences (Chudnovsky and Seymour, in preparation [2]).

In this paper, we prove and use a semigroup lemma to give a short proof of the bounded degree case of Rao's Conjecture that is independent of the Chudnovsky–Seymour structure theory. In fact, we affirmatively answer two questions of N. Robertson (2006) [8], the first of which implies the bounded degree case of Rao's Conjecture.

1. Introduction

Let G be a finite, simple graph and let $D(G) = (d_1, \ldots, d_n)$ be its sequence of vertex degrees listed in decreasing order. The sequence $D(G)$ is known as the *degree sequence* of G, and G is said to *realize* D. A sequence $(d_1, \ldots, d_n)$ of nonnegative integers is said to be a *graphic degree sequence* if it is realized by some graph. Given graphic degree sequences D_1 and D_2, we define $D_1 \preccurlyeq D_2$ to mean there is G_1 realizing D_1 and G_2 realizing D_2 such that $G_1 \sqsubseteq G_2$, where $\sqsubseteq$ is the induced subgraph relation. The reader may check that $\preccurlyeq$ is a transitive relation on degree sequences. For other basic graph theoretic definitions, we refer the reader to [3].

We recall that a *quasi order* $(Q, \leqslant)$ is a reflexive, transitive relation $\leqslant$ on a class Q. A quasi order $(Q, \leqslant)$ is said to be a *well-quasi order* if Q contains no infinite strictly decreasing sequence and no infinite antichain. Equivalently, $(Q, \leqslant)$ is a well-quasi order if for every infinite sequence $q_1, q_2, \ldots$ in Q there are positive integers $i < j$ such that $q_i \leqslant q_j$.

E-mail address: altomare@math.ohio-state.edu.

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With these definitions, we may state Rao's Conjecture, posed in 1980 by S.B. Rao [7] and finally proved in 2008 by M. Chudnovsky and P. Seymour [2].

Theorem 1. *Degree sequences of finite graphs are well quasi ordered by* $\preccurlyeq$.

Independently, N. Robertson had asked [8] if graphic degree sequences of bounded degree can be realized as disjoint unions of graphs with bounded sized components, noticing that an affirmative answer would imply the bounded degree case of Theorem 1. Motivated by this question, he further asked for a bipartite analogue. Namely, Robertson asked if degree sequences of bipartite graphs of bounded degree can be realized as disjoint unions of bipartite graphs with bounded sized components [8].

In this work, we use a semigroup lemma to prove that both of Robertson's questions have affirmative answers. In particular, we obtain a new proof of the bounded degree case of Theorem 1 that does not depend on the Chudnovsky–Seymour structure theory. This proof simplifies the proof given in the author's doctoral thesis [1].

2. The semigroup lemma

A *commutative semigroup* is a set S together with an associative, commutative binary operation $+$. We need not assume existence of an identity element. For basic facts and terminology, we refer the reader to [6], but our presentation is self contained. Given a semigroup $(S,+)$ and subsets $Y \subseteq X$ of S, we say that Y *generates* X if every x in X can be written as $y_1 + y_2 + \cdots + y_n$ for some points $y_1, \ldots, y_n$ of Y. We say that X is *finitely generated* if some finite subset Y of X generates X.

We now work exclusively with the free commutative semigroup $\mathbb{N}^k$, where we assume k is fixed throughout. Given $x = (x_1, \ldots, x_k)$ in $\mathbb{N}^k$, the *support* $\operatorname{supp}(x)$ is defined as the set of i such that $x_i > 0$.

Definition 2. Let X be a subset of $\mathbb{N}^k$. We say that X is *grounded* if for all i in $\{1, \ldots, k\}$ there is x in X with $\operatorname{supp}(x) = \{i\}$.

Let $(x_1, \ldots, x_k)$, $(y_1, \ldots, y_k)$, and $(t_1, \ldots, t_k)$ be elements in $\mathbb{N}^k$. Then

$$(x_1, \ldots, x_k) \equiv (y_1, \ldots, y_k) \mod (t_1, \ldots, t_k)$$

if $x_i \equiv y_i \bmod t_i$ for each i.

Lemma 3. *Every grounded subset of* $(\mathbb{N}^k, +)$ *is finitely generated.*

Proof. Fix a grounded set X. Then for each i in $\{1, \ldots, k\}$, we may choose an element of the form $(0, \ldots, 0, t_i, 0, \ldots, 0)$ in X, where $t_i > 0$ occurs in position i. Without loss of generality, we may choose the minimal such t_i for each i. Note that equivalence modulo $(t_1, \ldots, t_k)$ is an equivalence relation $\sim$ on $\mathbb{N}^k$ with only finitely many equivalence classes.

The partial order $(\mathbb{N}, \leqslant)$ with the usual ordering of the natural numbers is obviously a well-quasi order. Since the product of finitely many well-quasi orders is a well-quasi order, we see that the product order $(\mathbb{N}^k, \leqslant)$ is well quasi ordered. In particular, every antichain in $(\mathbb{N}^k, \leqslant)$ is finite.

Given a nonempty $\sim$ class C, the (possibly empty) set M_C of minimal elements of $C \backslash \{(0, \ldots, 0)\}$ is an antichain in $(\mathbb{N}^k, \leqslant)$ and therefore finite by the previous paragraph. Let Y be the union over all $\sim$ classes C of the sets M_C. Then Y is the finite union of finite sets and so is finite. It is enough to show Y generates X if $(0, \ldots, 0)$ is not in X, for then we know that $Y \cup (0, \ldots, 0)$ generates X if X includes $(0, \ldots, 0)$ as well.

Choose $x = (x_1, \ldots, x_k)$ in X. Let C be the $\sim$ class of x. Since $(\mathbb{N}^k, \leqslant)$ has no infinite strictly decreasing sequence, C contains an $(\mathbb{N}^k, \leqslant) - \{(0, \ldots, 0)\}$ minimal element $(m_1, \ldots, m_k)$ such that $(m_1, \ldots, m_k) \leqslant (x_1, \ldots, x_k)$. Then $m_i \leqslant x_i$ for each i. Since $(m_1, \ldots, m_k) \sim (x_1, \ldots, x_k)$ by hypothesis, we see that for each i, the equation $x_i - m_i = c_i t_i$ holds for some nonnegative integer c_i. Therefore

$$(x_1, \dots, x_k) = (m_1, \dots, m_k) + \sum_{i=1}^{k} c_i(0, \dots, 0, t_i, 0, \dots, 0).$$

We know $(m_1, \dots, m_k)$ is in Y by hypothesis. It is easy to see that $(0, \dots, 0, t_i, 0, \dots, 0)$ is a minimal nonzero element in its $\sim$ class. Therefore $(0, \dots, 0, t_i, 0, \dots, 0)$ is in Y as well, by which we see Y generates $(x_1, \dots, x_k)$. As $(x_1, \dots, x_k)$ in X was chosen arbitrarily, we see that Y generates X as claimed. $\square$

3. The main theorems

We now apply Lemma 3 to answer Robertson's original questions. Though Robertson asked if graphic degree sequences of bounded degree may be realized with bounded sized components, we note this is equivalent to asking if graphic degree sequences of bounded degree may be realized as disjoint unions of graphs from a fixed finite set, and similarly for the bipartite analogue. To see the first direction of this equivalence, note that for every fixed finite set F of graphs, the set of components of graphs in F has bounded size. For the other direction, note that if F is a set of graphs of bounded size, then the set of components of graphs in F is finite since there are only finitely many graphs up to isomorphism of any given finite cardinality. We find this reformulation somewhat more convenient as then Lemma 3 more directly applies.

Given a graph G, we define the *regularity sequence* R_G of G by letting $R_G(i) = |\{v \in V(G): d_G(v) = i\}|$ for each nonnegative integer i. If the graph has maximum degree at most k, then $R_G(i) = 0$ for $i > k$ and we may consider R_G as the point $(R_G(0), \dots, R_G(k))$ in $\mathbb{N}^{k+1}$. Given graphs G and H, we let $R_G \preccurlyeq R_H$ iff $D(G) \preccurlyeq D(H)$. Since degree and regularity sequences contain the same information, we freely switch between the two in proofs below.

We relate this to semigroup addition. If $G \oplus H$ denotes the disjoint union of graphs G and H, then $R_{G\oplus H} = R_G + R_H$. Given a nonnegative integer a and a graph G, we let aG denote the disjoint union of a copies of G. (If $a = 0$ then aG is the empty graph.) We let $\bigoplus_{i\in I} G_i$ denote the disjoint union of the graphs G_i for i in I.

Theorem 4. *Degree sequences of finite, bipartite graphs with bounded degree can be realized as disjoint unions of bipartite graphs from a fixed finite set F.*

Proof. We may instead show the same for regularity sequences, so let X be the set of regularity sequences of finite, bipartite graphs of degree at most k, considered as a subset of $\mathbb{N}^{k+1}$. To see that X is grounded, simply note that $K_{j,j}$ is a j-regular bipartite graph for each nonnegative j. We thus see by Lemma 3 that X is finitely generated. Let Y be a finite generating set. For each y in Y, there is a bipartite graph G_y with regularity sequence y. Since Y generates X, given an arbitrary finite, bipartite graph G, we see that $R_G = \sum_{y\in Y} a_y y$ for some nonnegative integers a_y. We then see that G has the same regularity and degree sequence as $\bigoplus_{y\in Y} a_y G_y$, which completes the proof. $\square$

The following theorem is that which Robertson proposed [8] as a stepping stone toward proving the bounded degree case of Rao's Conjecture.

Theorem 5. *Degree sequences of finite graphs with bounded degree can be realized as disjoint unions of graphs from a fixed finite set.*

Proof. The proof is exactly that of the previous theorem except we let X be the set of regularity sequences of finite graphs of degree at most k. $\square$

It is worth noting that neither Theorem 4 nor Theorem 5 is stronger than the other, as both the hypotheses and the conclusions of Theorem 4 are stronger than that of Theorem 5. We now give the simple proof that Theorem 5 implies the bounded degree case of Rao's Conjecture.

Corollary 6. *Fix k. Degree sequences of finite graphs with degrees at most k are well quasi ordered by $\preccurlyeq$.*

Proof. By Theorem 5 there is a finite set of graphs $\{G_1, G_2, \ldots, G_N\}$ that generates all degree sequences, or regularity sequences, for graphs with maximum degree at most k. Let R_G denote the regularity sequence of graph G. Then for any graphic regularity sequence $x \in \mathbb{N}^{k+1}$ there exists $a(x) \in \mathbb{N}^N$ such that $x = \sum_{i=1}^{N} a_i(x) R_{G_i}$. In general $a(x)$ is not unique, but we fix a unique $a(x)$ for each x. Now define an ordering $\sqsubseteq$ on $\mathbb{N}^{k+1}$ by $x \sqsubseteq x'$ if and only if $a(x) \leqslant a(x')$ in $\mathbb{N}^N$. Then it is clear that if $x \sqsubseteq x'$ we have $x \preccurlyeq x'$, so every $\preccurlyeq$-antichain is also a $\sqsubseteq$-antichain. But a $\sqsubseteq$-antichain corresponds to a $\leqslant$-antichain in $\mathbb{N}^N$, and is therefore finite; so all $\preccurlyeq$-antichains are finite, as required. $\square$

4. Conclusion

While our proof has the disadvantage of only going through for bounded degree, it is fairly short. Moreover, our proof is no longer restricted to graphs and goes through equally well for partial orders, hypergraphs, or any class of structured sets at all for which nonnegative integers can be assigned to each point in a way that respects disjoint union and such that "regular" elements exist. In particular, even for graphs, these nonnegative integers need no longer represent vertex degrees. This is worth noting since some of the most commonly used tools for degree sequences, such as switchings [5] and the Erdös–Gallai inequalities [4], have no known counterparts in this more general setting.

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